

Stochastic approach of gravitational waves in presence of a decaying cosmological parameter from a 5D vacuum

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We develop a stochastic approach to study gravitational waves produced during the inflationary epoch under the presence of a decaying cosmological parameter, on a 5D geometrical background which is Riemann flat. We obtain that the squared tensor metric fluctuations depend strongly on the cosmological parameter $\Lambda(t)$ and we finally illustrate the formalism with an example of a decaying $\Lambda(t)$.

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I. INTRODUCTION

Currently, embedding theorems are of rather interest in much due to their property of constraining the ways on which general relativity can be welded to higher dimensional scenarios, which may in principle embody internal symmetry groups like those that appear in particle physics. An interesting example of these are the Campbell-Magaard theorem and its known extensions that provide a kind of ladder between manifolds whose dimensionality differs just by one [1]. One of the most valuable physical implications of these theorems are that they guarantee that any physical source of 4D matter can be geometrically modeled from a Ricci-flat, Einstein or scalar field sourced higher dimensional manifolds. These powerful theorems make relevant the study of different cosmological topics on these settings. In particular to investigate in a stochastic approach the behavior of gravitational waves under the presence of a decaying cosmological parameter on a 5D Ricci-flat space-time can lead physics of vanguard in the study of cosmology of the early universe.

In the stochastic approach to inflation quantum to classical transition dynamics of the scalar field (inflaton) is effectively described by a classical noise, which has quantum origin [2, 3, 4]. This transition effect can be also studied by employing scalar metric fluctuations [5]. In this paper we investigate this transition dynamics in the case of linearized tensor perturbations [6], which as we know describe gravitational waves during inflation. Within the inflationary theory the prediction of the existence of a background of gravitational waves arises naturally [7]. These tensor perturbations escape out of the horizon during inflation, remaining this way completely conserved to form a relic of background gravitational waves, which carries out information of the very early universe [8, 9, 10]. This paper seeks to be a continuation of a recently introduced formalism where we have studied gravitational waves from a 5D vacuum state [11], considering an accelerated expansion of the universe governed by a decreasing cosmological parameter $\Lambda(t)$ during inflation [12]. Five dimensions are of particular interest, since this represents the simplest extension of spacetime and is widely regarded as the low-energy limit of even higher-dimensional theories with relevance to particle physics, such as a 10D supersymmetry, 11D supergravity and higher- D versions of string theory. However, our approach is inspired in the Induced Matter theory (STM) [15], where 4D sources appear as induced by one extended extra dimension, meaning, by extended, that the fifth dimension is considered noncompact. In the present approach the components of the tensor metric fluctuations are coarse-grained with an increasing number of degrees of freedom. In consequence the dynamics of the components of the coarse-grained tensor field are described by a set of second order stochastic equations, which can be rewritten as two sets of first order.

The paper is organized as follows: in Sect. II we introduce the formalism for tensor metric fluctuations on a 5D Riemann flat metric. In Sect. III we describe an effective 4D dynamics for these fluctuations when we take a

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foliation $\psi = \psi_0$ on the fifth coordinate. Furthermore, we describe the dynamics of the coarse-grained field, which describes the tensor fluctuations on super-Hubble scales, which are the relevant scales for cosmology during the inflationary evolution of the universe. In particular, we study the evolution of the squared $^{(L)}h$ -field fluctuations and its spectrum. Finally, in Sect. IV we develop some conclusions and final comments.

II. GW IN A 5D VACUUM: PRELIMINARY FORMALISM

In this section we shall establish the basic formalism with the necessary ingredients in order to construct a stochastic formalism able to describe GW from a 5D vacuum in the presence of a time dependent cosmological parameter $\Lambda(t)$. With this aim we start considering the background line element [16]

$$dS_b^2 = \psi^2 \frac{\Lambda(t)}{3} dt^2 - \psi^2 e^2 \int \sqrt{\Lambda/3} dt dr^2 - d\psi^2, \quad (1)$$

where $dr^2 = \delta_{ij} dx^i dx^j$, being $\{x^i\} = \{x, y, z\}$ the local cartesian coordinates. Here t is the cosmic time and ψ is the fifth coordinate which is space-like. Adopting a natural unit system (where $\hbar = c = 1$), the fifth coordinate ψ has spatial units whereas the cosmological parameter $\Lambda(t)$ has units of $(length)^{-2}$. The background metric in (1) is Riemann-flat, $R^A{}_{BCD} = 0$ and thereby it describes a 5D geometrical vacuum.

The second order 5D action for the tensor perturbations in our case is

$$^{(5)}\mathcal{S} = \int d^4x d\psi \sqrt{\left| \frac{^{(5)}g}{^{(5)}g_0} \right|} \left[\frac{^{(5)}\mathcal{R}}{16\pi G} + \frac{M_p^2}{2} g^{AB} Q_{;A}^{ij} Q_{ij;B} \right], \quad (2)$$

being $Q_{ij}(t, \vec{r}, \psi)$ the transverse traceless tensor denoting the tensor fluctuations with respect to the background metric g_{AB} , and therefore the expressions $tr(Q_{ij}) = Q^i{}_i = 0$ and $Q^{ij}{}_{;i} = 0$ are valid. In addition the comma (;) is denoting covariant derivative and $^{(5)}g_0 = ^{(5)}g[\psi = \psi_0, \Lambda_0 = \Lambda(t = t_0)]$ is a dimensionalization constant being ψ_0 and t_0 some constants to be specified. From the action (2) we derive the evolution equation for the tensor fluctuations Q_{ij}

$$\ddot{Q}_{ij} + \left[3\sqrt{\frac{\Lambda}{3}} - \frac{1}{2} \frac{\dot{\Lambda}}{\Lambda} \right] \dot{Q}_{ij} - \frac{\Lambda}{3} e^{-2 \int \sqrt{\Lambda/3} dt} \nabla_r^2 Q_{ij} - \frac{\Lambda}{3} [4\psi Q_{ij,\psi} + \psi^2 Q_{ij,\psi,\psi}] = 0, \quad (3)$$

where the dot denotes derivative with respect to the cosmic time t . Quantization of Q_{ij} is achieved by demanding the commutation relation

$$\left[Q_{ij}(t, \vec{r}, \psi), \frac{\partial ^{(5)}L^{(GW)}}{\partial Q_{ls,t}}(t, \vec{r}, \psi) \right] = i \delta_i^l \delta_j^s g^{tt} M_p^2 \sqrt{\left| \frac{^{(5)}g}{^{(5)}g_0} \right|} \left(\frac{\psi_0}{\psi} \right)^3 e^{-\int [3\sqrt{\frac{\Lambda}{3}} - \frac{\dot{\Lambda}}{2\Lambda}] dt} \delta^{(3)}(\vec{r} - \vec{r}'), \quad (4)$$

and expressing the quantum operators Q_{ij} as a Fourier expansion of the form

$$Q^i{}_j(t, \vec{r}, \psi) = \frac{1}{(2\pi)^{3/2}} \int d^3k_r \sum_{\alpha} ^{(\alpha)} e^i{}_j \left[e^{i\vec{k}_r \cdot \vec{r}} \zeta_{\vec{k}_r}^{(\alpha)}(t, \psi) + e^{-i\vec{k}_r \cdot \vec{r}} \left(\zeta_{\vec{k}_r}^{(\alpha)}(t, \psi) \right)^* \right], \quad (5)$$

with α counting the number of polarization degrees of freedom and the asterisk (*) denoting complex conjugate. The polarization tensor $^{(\alpha)}e_{ij}$ obeys

$$^{(\alpha)}e_{ij} = ^{(\alpha)}e_{ji}, \quad ^{(\alpha)}e_{ii} = 0, \quad k^i ^{(\alpha)}e_{ij} = 0, \quad ^{(\alpha)}e_{ij}(-\vec{k}_r) = ^{(\alpha)}e_{ij}^*(\vec{k}_r). \quad (6)$$

On the other hand, by introducing the quantities

$$\chi_{ij}(t, \vec{r}, \psi) = e^{\frac{1}{2} \int [3\sqrt{\frac{\Lambda}{3}} - \frac{\dot{\Lambda}}{2\Lambda}] dt} Q_{ij}(t, \vec{r}, \psi), \quad (7)$$

the evolution equation (3) becomes

$$\ddot{\chi}^i{}_j - \frac{\Lambda}{3} e^{-2 \int \sqrt{\Lambda/3} dt} \nabla_r^2 \chi^i{}_j + \left[\frac{1}{4} \frac{\ddot{\Lambda}}{\Lambda} - \frac{3}{4} \Lambda - \frac{5}{16} \frac{\dot{\Lambda}^2}{\Lambda^2} \right] \chi^i{}_j - \frac{\Lambda}{3} \left[4\psi \frac{\partial}{\partial \psi} + \psi^2 \frac{\partial^2}{\partial \psi^2} \right] \chi^i{}_j = 0. \quad (8)$$

Inserting the expansion (5) in (7) we obtain the Fourier expansion for the quantum operators χ^i_j

$$\chi^i_j(t, \vec{r}, \psi) = \frac{1}{(2\pi)^{3/2}} \int d^3k_r \sum_{\alpha} {}^{(\alpha)}e^i_j \left[e^{i\vec{k}_r \cdot \vec{r}} \xi_{k_r}^{(\alpha)}(t, \psi) + e^{-i\vec{k}_r \cdot \vec{r}} \left(\xi_{k_r}^{(\alpha)}(t, \psi) \right)^* \right], \quad (9)$$

where we have introduced the re-defined modes

$$\xi_{k_r}^{(\alpha)}(t, \psi) = e^{\frac{1}{2} \int [3\sqrt{\frac{\Lambda}{3}} - \frac{\Lambda}{2\Lambda}] dt} \zeta_{k_r}^{(\alpha)}(t, \psi). \quad (10)$$

The dynamical equation for the re-defined modes then reads

$$\ddot{\xi}_{k_r}^{(\alpha)} + \left[\frac{\Lambda}{3} e^{-2 \int \sqrt{\Lambda/3} dt} k_r^2 + \frac{1}{4} \frac{\ddot{\Lambda}}{\Lambda} - \frac{3}{4} \Lambda - \frac{5}{16} \frac{\dot{\Lambda}^2}{\Lambda^2} \right] \xi_{k_r}^{(\alpha)} - \frac{\Lambda}{3} \left[4\psi \frac{\partial}{\partial \psi} + \psi^2 \frac{\partial^2}{\partial \psi^2} \right] \xi_{k_r}^{(\alpha)} = 0. \quad (11)$$

The equation (11) yields

$$\ddot{\xi}_{k_r} + \left[\frac{\Lambda}{3} e^{-2 \int \sqrt{\Lambda/3} dt} k_r^2 - \frac{1}{4} \sqrt{\frac{3}{\Lambda}} \dot{\Lambda} + \frac{1}{4} \frac{\ddot{\Lambda}}{\Lambda} - \frac{5}{16} \frac{\dot{\Lambda}^2}{\Lambda^2} + \frac{3}{4} \sqrt{\frac{\Lambda}{3}} \frac{\dot{\Lambda}}{\Lambda} + \left(\frac{m^2}{3} - \frac{3}{4} \right) \Lambda \right] \xi_{k_r} = 0, \quad (12)$$

$$\frac{d^2 L_m(z)}{dz^2} + \left[m^2 - \frac{9}{4} \right] L_m(z) = 0, \quad (13)$$

where we have used the transformation $\Theta_m(z) = e^{-(3/2)z} L_m(z)$ with $z = \ln(\psi/\psi_0)$ and the parameter m^2 is a separation constant which is related with the squared of the KK mass measured by a class of 5D observers. This way, given a cosmological parameter $\Lambda(t)$, the temporal evolution of the tensor modes $\xi_{k_r}(t)$ in 5D is determined by solutions of (12). Once solutions for $\xi_{k_r}(t)$ and $L_m(z)$ are obtained, they should satisfy the algebra (4). This can be achieved if the solutions satisfy respectively the condition

$$\int d^3k_r \left\langle 0 \left| \left[\xi_{k_r}^{(\alpha)} \left(\dot{\xi}_{k_r}^{(\alpha)} \right)^* - \dot{\xi}_{k_r}^{(\alpha)} \left(\xi_{k_r}^{(\alpha)} \right)^* \right] \right| 0 \right\rangle = i, \quad (14)$$

which are usually named the normalization conditions.

On the other hand, note that equation (13) is exactly the same as the one obtained in [11]. Therefore about the behavior of the modes with respect the fifth coordinate we can say that for $m > 3/2$ the KK-modes are coherent on the ultraviolet sector (UV), described by the modes

$$k_r^2 > k_0^2(t) = \left\{ \frac{3}{2\Lambda} \frac{d}{dt} \left[3\sqrt{\frac{\Lambda}{3}} - \frac{\dot{\Lambda}}{2\Lambda} \right] - \frac{3}{4\Lambda} \left(3\sqrt{\frac{\Lambda}{3}} - \frac{\dot{\Lambda}}{2\Lambda} \right)^2 - m^2 \right\} e^{2 \int \sqrt{\Lambda/3} dt} > 0. \quad (15)$$

Notice that in the case of a constant cosmological parameter $\Lambda = \Lambda_0$, we obtain that

$$k_0^2(t) = \left(\frac{9}{4} - m^2 \right) e^{2 \int \sqrt{\Lambda/3} dt},$$

and the line element (1) give us the Ponce de León metric[14]. Notice that for $m > 3/2$ the solutions of (12) are always stable. However, for $m < 3/2$ those modes are unstable and diverge at infinity when $k_r < k_0$. The modes with $m > 3/2$ comply with the conditions (14), so that they are normalizable.

The coarse-grained tensor field which describes a stochastic dynamics on super-Hubble scales is defined by

$$\chi^i_j(t, \vec{r}, \psi) = \frac{1}{(2\pi)^{3/2}} \int d^3k_r \theta(\epsilon k_0 - k_r) \sum_{\alpha} {}^{(\alpha)}e^i_j \left[e^{i\vec{k}_r \cdot \vec{r}} \xi_{k_r}^{(\alpha)}(t, \psi) + e^{-i\vec{k}_r \cdot \vec{r}} \left(\xi_{k_r}^{(\alpha)}(t, \psi) \right)^* \right], \quad (16)$$

This field ${}^{(L)}\chi^i_j(t, \vec{r}, \psi)$ contains all the modes in the IR-sector $k_r/k_0 < \epsilon \simeq 10^{-3}$. This means that ${}^{(L)}\chi^i_j$ only considers modes with wavelengths larger than 10^3 times the size of the horizon during inflation.

III. EFFECTIVE 4D DYNAMICS

As in a recently introduced work [12] we shall assume that the 5D space-time can be foliated by a family of hypersurfaces where a generic hypersurface is determined by taking $\psi = \psi_0$. An extension to dynamical foliations was recently studied in [13]. Thus, the line element (1) generates an effective 4D background metric $dS^2|_{eff} = ds^2$, where

$$ds^2 = \psi_0^2 \frac{\Lambda(t)}{3} dt^2 - \psi_0^2 e^{2 \int \sqrt{\frac{\Lambda}{3}} dt} d\vec{r}^2. \quad (17)$$

It is important to notice that when the cosmological parameter is constant, $\Lambda = \Lambda_0$, the effective 4D metric describes a de Sitter expansion with an energy density $\rho_v = \frac{\Lambda_0}{8\pi G} = -p_v$, where ρ_v and p_v are respectively the energy density and the pressure on a vacuum dominated expansion. In particular, when we use the foliation $\psi_0 = \sqrt{3/\Lambda_0} = H_0^{-1}$ (in this case H_0 is the constant Hubble parameter), we obtain a comoving reference system with tetra-velocities $u^\alpha = (1, 0, 0, 0)$ and the universe can be described by a Friedmann-Robertson-Walker (FRW) metric $ds^2 = dt^2 - a^2(t) d\vec{r}^2$ [$a^2 = H_0^{-2} e^{2H_0 t}$] with an exponential (vacuum dominated) expansion. In general [i.e., when $\Lambda = \Lambda(t)$], the effective 4D metric (17) is not the usual FRW one, and the tetra-velocities u^t and u^r are related by the expression

$$(u^t)^2 = \frac{3}{\Lambda} \left[\psi_0^{-2} + (u^r)^2 e^{-2 \int \sqrt{\Lambda/3} dt} \right],$$

where $(u^r)^2 = (u^x)^2 + (u^y)^2 + (u^z)^2$.

On the other hand, the dynamics of the 4D tensor-fluctuations will be given in terms of the tensor components $h_{ij}(t, \vec{r}) \equiv Q_{ij}(t, \vec{r}, \psi = \psi_0)$. The effective 4D action (α, β run from 0 to 3) can be written as

$${}^{(4)}\mathcal{S} = - \int d^4x \sqrt{\left| \frac{{}^{(4)}g}{{}^{(4)}g_0} \right|} \left[\frac{{}^{(4)}\mathcal{R}}{16\pi G} + \frac{M_p^2}{2} g^{\alpha\beta} Q_{;\alpha}^{ij} Q_{ij;\beta} \right] \Bigg|_{\psi=\psi_0}, \quad (18)$$

where ${}^{(4)}\mathcal{R} = 12/\psi_0^2$ is the effective 4D Ricci scalar evaluated on the metric (17). In other words, the 4D scalar curvature is geometrically induced by the foliation on the fifth coordinate: $\psi = \psi_0$.

The effective 4D linearized equation of motion for the 4D tensor-fluctuations is

$$\ddot{h}_j^i + \left[3\sqrt{\frac{\Lambda}{3}} - \frac{\dot{\Lambda}}{2\Lambda} \right] \dot{h}_j^i - \frac{\Lambda}{3} e^{-2 \int \sqrt{\frac{\Lambda}{3}} dt} \nabla_r^2 h_j^i + \frac{\Lambda}{3} \left[4\psi \frac{\partial}{\partial \psi} + \psi^2 \frac{\partial^2}{\partial \psi^2} \right] h_j^i \Bigg|_{\psi=\psi_0} = 0, \quad (19)$$

which, after make the transformation $h_j^i(t, \vec{r}) = e^{-1/2 \int [3(\frac{\Lambda}{3})^{1/2} - \frac{\dot{\Lambda}}{2\Lambda}] dt} \chi_j^i(t, \vec{r})$, give us the equation of motion for the redefined 4D tensor-fluctuations

$$\ddot{\chi}_j^i - \frac{\Lambda}{3} e^{-2 \int \sqrt{\frac{\Lambda}{3}} dt} \nabla_r^2 \chi_j^i - \left[\frac{1}{4} \sqrt{\frac{3}{\Lambda}} \dot{\Lambda} + \frac{1}{4} \frac{\ddot{\Lambda}}{\Lambda} - \frac{5}{16} \frac{\dot{\Lambda}^2}{\Lambda^2} + \frac{3}{4} \sqrt{\frac{\Lambda}{3}} \frac{\dot{\Lambda}}{\Lambda} + \left[\frac{1}{3} \left(4\psi \frac{\partial}{\partial \psi} + \psi^2 \frac{\partial^2}{\partial \psi^2} \right) - \frac{3}{4} \right] \Lambda \right] \chi_j^i \Bigg|_{\psi=\psi_0} = 0. \quad (20)$$

Therefore, for a given $\Lambda(t)$ we can obtain in principle an effective 4D dynamics for the tensor fluctuations of the metric. Now, instead of following the standard procedure to investigate the 4D effective dynamics of this tensor modes, let us to adopt a stochastic approach.

A. Coarse grained field in 4D

In order to describe the tensor-fluctuations on cosmological scales, we shall introduce the coarse-grained tensor field ${}^{(L)}\chi_j^i(t, \vec{r}, \psi = \psi_0)$ on the effective 4D metric (17). This field is represented as a Fourier expansion on the modes whose wavelengths are bigger than the Hubble radius

$$\begin{aligned} {}^{(L)}\chi_j^i(t, \vec{r}, \psi_0) &= \frac{1}{(2\pi)^{3/2}} \int d^3k_r \theta(\epsilon k_0 - k_r) \sum_{\alpha} {}^{(\alpha)}e^i{}_j \\ &\times \left[e^{i\vec{k}_r \cdot \vec{r}} \xi_{k_r}^{(\alpha)}(t, \psi_0) + e^{-i\vec{k}_r \cdot \vec{r}} \left(\xi_{k_r}^{(\alpha)}(t, \psi_0) \right)^* \right], \end{aligned} \quad (21)$$

where, because on the effective 4D hypersurface ψ is a constant ψ_0 , we shall consider that m is a free parameter, such that

$$\xi_{k_r}^{(\alpha)}(t, \psi_0) = a_{k_r}^{(\alpha)} \xi_{k_r}(t), \quad \left[a_{k_r}^{(\alpha)}, a_{k_r'}^{(\alpha') \dagger} \right] = g^{\alpha\alpha'} \delta^{(3)}(\vec{k}_r - \vec{k}_r'), \quad (22)$$

and

$$k_0(t) = \left[\frac{5}{16} \frac{\dot{\Lambda}^2}{\Lambda^2} - \frac{1}{4} \frac{\ddot{\Lambda}}{\Lambda} + \left(\frac{3}{4} - \frac{m^2}{3} \right) \Lambda \right]^{1/2} \sqrt{\frac{3}{\Lambda}} e^{\int \sqrt{\frac{\Lambda}{3}} dt}. \quad (23)$$

The equation of motion for the field ${}^{(L)}\chi^i{}_j(t, \vec{r}, \psi_0)$ is

$${}^{(L)}\ddot{\chi}^i{}_j - \frac{\Lambda}{3} k_0^2 e^{-2 \int \sqrt{\frac{\Lambda}{3}} dt} {}^{(L)}\chi^i{}_j = \epsilon \left[\ddot{k}_0 \eta^i{}_j(t, \vec{r}, \psi_0) + \dot{k}_0 \kappa^i{}_j(t, \vec{r}, \psi_0) + 2\dot{k}_0 \gamma^i{}_j(t, \vec{r}, \psi_0) \right], \quad (24)$$

where the stochastic tensor operators $\eta^i{}_j$, $\kappa^i{}_j$ and $\gamma^i{}_j$, on the effective 4D metric (17), are given, respectively, by

$$\begin{aligned} \eta^i{}_j(t, \vec{r}, \psi_0) &= \frac{1}{(2\pi)^{3/2}} \int d^3 k_r \delta(\epsilon k_0 - k_r) \sum_{\alpha} {}^{(\alpha)} e^i{}_j \\ &\times \left[e^{i\vec{k}_r \cdot \vec{r}} \xi_{k_r}^{(\alpha)}(t, \psi_0) + e^{-i\vec{k}_r \cdot \vec{r}} \left(\xi_{k_r}^{(\alpha)}(t, \psi_0) \right)^* \right], \end{aligned} \quad (25)$$

$$\begin{aligned} \kappa^i{}_j(t, \vec{r}, \psi_0) &= \frac{1}{(2\pi)^{3/2}} \int d^3 k_r \delta(\epsilon k_0 - k_r) \sum_{\alpha} {}^{(\alpha)} e^i{}_j \\ &\times \left[e^{i\vec{k}_r \cdot \vec{r}} \xi_{k_r}^{(\alpha)}(t, \psi_0) + e^{-i\vec{k}_r \cdot \vec{r}} \left(\xi_{k_r}^{(\alpha)}(t, \psi_0) \right)^* \right], \end{aligned} \quad (26)$$

$$\begin{aligned} \gamma^i{}_j(t, \vec{r}, \psi_0) &= \frac{1}{(2\pi)^{3/2}} \int d^3 k_r \delta(\epsilon k_0 - k_r) \sum_{\alpha} {}^{(\alpha)} e^i{}_j \\ &\times \left[e^{i\vec{k}_r \cdot \vec{r}} \dot{\xi}_{k_r}^{(\alpha)}(t, \psi_0) + e^{-i\vec{k}_r \cdot \vec{r}} \left(\dot{\xi}_{k_r}^{(\alpha)}(t, \psi_0) \right)^* \right]. \end{aligned} \quad (27)$$

By using differential properties of the former stochastic operators, the equation (24) can be written as

$${}^{(L)}\ddot{\chi}^i{}_j - \frac{\Lambda}{3} k_0^2 e^{-2 \int \sqrt{\frac{\Lambda}{3}} dt} {}^{(L)}\chi^i{}_j = \epsilon \left[\frac{\partial}{\partial t} \left(\dot{k}_0 \eta^i{}_j(t, \vec{r}, \psi_0) \right) + \dot{k}_0 \gamma^i{}_j(t, \vec{r}, \psi_0) \right]. \quad (28)$$

This is a second-order stochastic equation that can be written as a first-order system in the form

$$\dot{u}^i{}_j = \frac{\Lambda}{3} k_0^2 e^{-2 \int \sqrt{\frac{\Lambda}{3}} dt} {}^{(L)}\chi^i{}_j + \epsilon \dot{k}_0 \gamma^i{}_j, \quad (29)$$

$${}^{(L)}\dot{\chi}^i{}_j = u^i{}_j + \epsilon \dot{k}_0 \eta^i{}_j, \quad (30)$$

where we have introduced the auxiliary field $u^i{}_j \equiv {}^{(L)}\dot{\chi}^i{}_j - \epsilon \dot{k}_0 \eta^i{}_j$. Now, in order to minimize the role of the stochastic noise $\gamma^i{}_j$, we impose the condition $\dot{k}_0^2 \langle \gamma^2 \rangle \ll \dot{k}_0^2 \langle \eta^2 \rangle$, where we have defined the quantities $\langle \gamma^2 \rangle = \langle 0 | \gamma^i{}_j \gamma^j{}_i | 0 \rangle$ and $\langle \eta^2 \rangle = \langle 0 | \eta^i{}_j \eta^j{}_i | 0 \rangle$. This condition can be expressed in terms of the modes as

$$\left. \frac{\dot{\xi}_{k_r}^{(\alpha)} \left(\dot{\xi}_{k_r}^{(\alpha)} \right)^*}{\xi_{k_r}^{(\alpha)} \left(\xi_{k_r}^{(\alpha)} \right)^*} \right|_{\psi=\psi_0} \ll \left(\frac{\ddot{k}_0}{\dot{k}_0} \right)^2, \quad (31)$$

which only is valid on super-Hubble scales. Under this consideration the system (29)-(30) becomes

$$\dot{u}^i{}_j = \frac{\Lambda}{3} k_0^2 e^{-2 \int \sqrt{\frac{\Lambda}{3}} dt} {}^{(L)}\chi^i{}_j, \quad (32)$$

$${}^{(L)}\dot{\chi}^i{}_j = u^i{}_j + \epsilon \dot{k}_0 \eta^i{}_j. \quad (33)$$

This new system can be seen as two first-order Langevin equations with a tensor noise $\eta^i{}_j$ which is Gaussian and white in nature. Hence, it satisfies

$$\langle \eta \rangle = \langle g^j{}_i \eta^i{}_j \rangle = 0, \quad (34)$$

$$\langle \eta^2 \rangle|_{\psi=\psi_0} = \langle \eta^i{}_j \eta^j{}_i \rangle|_{\psi=\psi_0} = \frac{3\epsilon k_0^2}{\pi^2 \dot{k}_0} \xi_{\epsilon k_0}(t) \xi_{\epsilon k_0}^*(t) \delta(t - t'). \quad (35)$$

The corresponding Fokker-Planck equation that describes the dynamics of the transition probability $\mathcal{P}^i_j[(^{(L)}\chi_0^i, u_0^i) | (^{(L)}\chi^i, u^i)]$ from a configuration $(^{(L)}\chi_0^i, u_0^i)$ to $(^{(L)}\chi^i, u^i)$ is then

$$\frac{\partial \mathcal{P}^i_j}{\partial t} = -u^i_j \frac{\partial \mathcal{P}^i_j}{\partial (^{(L)}\chi^i_j)} - \mu^2(t) (^{(L)}\chi^i_j) \frac{\partial \mathcal{P}^i_j}{\partial u^i_j} + \frac{1}{6} D_{\eta\eta} \frac{\partial^2 \mathcal{P}^i_j}{\partial (^{(L)}\chi^i_j)^2}, \quad (36)$$

where $\mu^2(t) = (\Lambda/3)k_0^2 \exp[-2 \int \sqrt{\Lambda/3} dt]$ and the only nonzero component of the diffusion tensor is $D_{\eta\eta}(t) = [(\epsilon \dot{k}_0^2)/2] \int dt \langle \eta^2 \rangle$. Note that we have considered that $D_{\eta\eta} = 3D_{\eta^i_j \eta^j_i}$, due to the 3D space $r(x, y, z)$ is isotropic. This diffusion coefficient is related to the field $(^{(L)}\chi)$ due to the stochastic action of the effective noise η (related to η^i_j). The equation of motion for $\langle (^{(L)}\chi^2) \rangle \equiv \langle 0 | (^{(L)}\chi^i_j) (^{(L)}\chi_i^j) | 0 \rangle = \int d(^{(L)}\chi) du (^{(L)}\chi) u \mathcal{P}[(^{(L)}\chi, u)]$, takes the form

$$\frac{d}{dt} \langle (^{(L)}\chi^2) \rangle = \frac{1}{2} D_{\eta\eta}(t), \quad (37)$$

where we have considered $g^j_i \mathcal{P}^i_j = \mathcal{P}$, $\chi = g^j_i \chi^i_j$ and $u = g^j_i u^i_j$. Therefore, the stochastic dynamics of $(^{(L)}\chi)$ is completely determined and consequently the corresponding evolution of $\langle (^{(L)}h^2) \rangle$ is given by the solution of

$$\frac{d}{dt} \langle (^{(L)}h^2) \rangle = - \left[3\sqrt{\frac{\Lambda}{3}} - \frac{\dot{\Lambda}}{2\Lambda} \right] \langle (^{(L)}h^2) \rangle + \frac{1}{2} e^{-\int dt [3\sqrt{\frac{\Lambda}{3}} - \frac{\dot{\Lambda}}{2\Lambda}]} D_{\eta\eta}(t). \quad (38)$$

The general solution of this equation is

$$\langle (^{(L)}h^2) \rangle = \frac{1}{2} e^{-\int dt [3\sqrt{\frac{\Lambda}{3}} - \frac{\dot{\Lambda}}{2\Lambda}]} \int D_{\eta\eta}(t) dt \quad (39)$$

where $D_{\eta\eta} = [3\epsilon^3 k_0^2 / (2\pi^2)] \xi_{\epsilon k_0}(t) \xi_{\epsilon k_0}^*(t)$. Therefore, for a given cosmological parameter $\Lambda(t)$, the stochastic squared fluctuations of h on super Hubble scales, $(^{(L)}h)$, is determined by (39).

B. An example

Now let us to illustrate the previous formalism by considering a decaying cosmological parameter on the metric (17): $\Lambda(t) = 3p^2/t^2$ (with $p > 0$), which clearly satisfies $\dot{\Lambda} < 0$. In this case the dynamical field equation for the modes $\xi_{k_r}(t)$ reads

$$\ddot{\xi}_{k_r} + \left\{ k_r^2 p^2 t_0^{2p} t^{-2(p+1)} - \left[\left(m^2 - \frac{9}{4} \right) p^2 - \frac{9}{4} p + \frac{1}{4} \right] t^{-2} \right\} \xi_{k_r} = 0, \quad (40)$$

where $M^2(t) = [(m^2 - 9/4)p^2 - (9/4)p + 1/4]t^{-2}$ can be interpreted as an effective squared term of mass. The permitted values for m should be $9/4 < m^2 \leq (9/4)[1 + 9/4]$, for which the next relation is valid

$$0 < p \leq \frac{9 + \sqrt{117 - 16m^2}}{2(4m^2 - 9)}. \quad (41)$$

However, as it was shown in [12], even when the general solution of (40) is not normalizable, there exist some particular normalizable solutions. One class of these solutions is obtained by considering $M^2(t) \geq 0$. In this case the expression (40) reduces to

$$\ddot{\xi}_{k_r} + \left[k_r^2 p^2 t_0^{2p} t^{-2(p+1)} - M^2(t) \right] \xi_{k_r} = 0, \quad (42)$$

whose normalized solution (using the Bunch-Davies vacuum[17]), is given by

$$\xi_{k_r}(t) = \frac{i}{2} \sqrt{\frac{1}{\pi p}} \sqrt{t} \mathcal{H}_\nu^{(2)} \left[k_r \left(\frac{t_0}{t} \right)^p \right] \quad (43)$$

where $\mathcal{H}_\nu^{(2)}$ is the second kind Hankel function and $\nu^2 = m^2 - [9p(p+1) - 2]/(4p^2)$. Thus considering that in this particular case $k_0(t) = \sqrt{\alpha/p} (t/t_0)^p$ and using the asymptotic expansion $\mathcal{H}_\nu^{(2)}[x] \simeq (-i/\pi) \Gamma(\nu) [x/2]^{-\nu}$, the diffusion coefficient $D_{\eta\eta}$ has the form

$$D_{\eta\eta}(t) = \frac{3\epsilon^{3-2\nu}}{\pi^5 p^{3/2-\nu}} 2^{-(3-2\nu)} \alpha^{3/2-\nu} \Gamma^2(\nu) \left(\frac{t}{t_0} \right)^{3p}, \quad (44)$$

where $\alpha = M^2(t)t^2/p^2 \geq 0$ is a real constant. Hence equation (39) gives

$$\langle {}^{(L)}h^2 \rangle = \frac{3\epsilon^{3-2\nu}}{\pi^5(3p+1)p^{3/2-\nu}} \Gamma^2(\nu) 2^{-4+2\nu} \alpha^{3/2-\nu} t_0 \left[1 - \left(\frac{t_0}{t} \right)^{1+3p} \right], \quad (45)$$

which, for $p > 0$ and $t > t_0$ is always a positive quantity. Note that when $\alpha = 0$ automatically $\langle {}^{(L)}h^2 \rangle = 0$. On the other hand, for a scale invariant spectrum (for which $\nu = 3/2$) we have

$$\langle {}^{(L)}h^2 \rangle = \frac{3\Gamma^2(\nu)}{2\pi^5(3p+1)} t_0 \left[1 - \left(\frac{t_0}{t} \right)^{1+3p} \right], \quad (46)$$

which is independent of the value of α . In this case we have that $9p^2 - 4\alpha p - 1 = 0$, so that $p = \frac{2\alpha}{9} \left[1 + \sqrt{1 + \frac{9}{4\alpha^2}} \right]$, for $\alpha > 0$.

In general, the power spectrum of $\langle {}^{(L)}h^2 \rangle$ has the form

$$\mathcal{P}_{\langle {}^{(L)}h^2 \rangle} \sim k_r^{n-1} = k_r^{3-2\nu}. \quad (47)$$

Recent calculations [18] showed that $n \simeq 1.2$, which corresponds to $\nu \simeq 1.4$. With this result we obtain

$$m^2 \simeq (1.4)^2 + \frac{[9p(p+1)-2]}{4p^2}, \quad (48)$$

which is the main result of this work. For $p > 1$, and $\nu \simeq 1.4$, we obtain the following restrictions for m :

$$2.052 < m < 2.44. \quad (49)$$

IV. FINAL REMARKS

In this letter we have developed a stochastic approach to study gravitational waves produced during inflation. We have considered that the expansion is governed by a decaying cosmological parameter. The formalism was constructed by considering a 5D geometrical background which is Riemann flat. Hence, all effective 4D sources are induced from the foliation $\psi = \psi_0$, which is taken on this 5D flat space-time. In our particular case, the large scale tensor metric fluctuations are linearized, so that they obey a like-wave equation of motion. Their components can be considered on the infrared sector (super Hubble or large scale tensor fluctuations ${}^{(L)}h^i{}_j$), which obey a set of stochastic equations, affected by tensor noises $\eta^i{}_j$ (gaussian and white in our case, due to the fact we have used a Heaviside function as a window function on ${}^{(L)}h^i{}_j$). Due to the isotropy of the 3D space, it is possible to define an effective diffusion coefficient $D_{\eta\eta}$ to describe the evolution of $\langle {}^{(L)}h^2 \rangle$. In particular, we obtain that for $\Lambda(t) = 3p^2/t^2$ ($p > 1$ and $m = \pm[1/(2p)]\sqrt{9p(p+1)+4\alpha p-1}$), the parameter α is restricted by the inequality $\frac{9p^2-1}{4p} = \alpha > 0$, for a scale invariant power spectrum of $\langle {}^{(L)}h^2 \rangle$. In general, the relevant result of our formalism is that the KK-mass results to be related to the power of the $\langle {}^{(L)}h^2 \rangle$ -spectrum, $m^2 \simeq n^2 + \frac{[9p(p+1)-2]}{4p^2}$, through the spectral index n and the parameter p that characterizes the decaying cosmological parameter $\Lambda(t)$.

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